ISSN : 2348-0033 (Online) | ISSN : 2249-4944 (Print)

Partial Differential Equation by Using Haar Wavelet Operational Method

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Abstract
In this paper, we use a method based on the operational matrices to the solution of the fractional partial differential equations. The main approach is based on the operational matrices of the Haar wavelets to obtain the algebraic equations. The fractional derivatives are described in Caputo sense. Some examples are included to demonstrate the validity and applicability of the techniques.

Keywords
Operational Matrix, Fractional Partial Differential Equation, Haar wavelets, Numerical Method.

I. Introduction

In recent years, fractional calculus is one of the interest issues that attract many scientists, specially mathematics and engineering sciences. Many natural phenomena can be present by boundary value problems of fractional differential equations. Many authors in different fields such as chemical physics, fluid flows, electrical networks, viscoelasticity, try to modeling of these phenomena by boundary value problems of fractional deferential equations [1-4]. For achieve extra information in fractional calculus, reader can refer to more valuable books that are written by authors [5-9]. Many physical phenomena in areas such as damping laws, diffusion processes, etc, can be modeled with partial differential equations of fractional order. Many authors developed wavelet-based numerical solutions of partial differential equations [10-16]. Useful applications of wavelet operational matrix for numerical solutions of differential equations can be found [17-23]. This issue with characteristic of wavelet functions is motivations for using them to find the solutions of differential equations of fractional order and mostly fractional partial differential equations. In this paper, we use a method based on the operational matrices to the solution of the fractional partial differential equations. The main approach is based on the operational matrices of the Haar wavelets to obtain the algebraic equations.

Why Wavelets?

- Efficiency to compress the smooth data expect in localized region.
- Good approximatrion properties.
- Space representation of calDERON-zYGMOND type operators.
- Easy to control wavlet Properties.

II. Haar Wavelets Operational Matrixes

The operational matrix of an orthogonal matrix $\Phi(t)$, F_Φ, can be expressed by

$$F_\Phi = \Phi \cdot F_B \cdot \Phi^{-1} \qquad (1)$$

Where F_B is the operational matrix of the block pulse function

$$F_{B_m} = \frac{1}{m}\begin{bmatrix} \frac{1}{2} & 1 & . & . & . & 1 \\ 0 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 1 & . \\ . & . & . & 0 & \frac{1}{2} & 1 \\ 0 & . & . & . & 0 & \frac{1}{2} \end{bmatrix} \qquad (2)$$

If the transform matrix φ is unitary, that is $\varphi^{-1} = \varphi^T$ then (1) can be rewrtten as:

$$F_\Phi = \Phi \cdot F_B \cdot \Phi^T \qquad (3)$$

The approach is simple and computer oriented, therefore very useful in practice. Wavelets have become an increasingly popular tool in the computational sciences. They have numerous applications in a wide range of areas such as signal analysis, data compression and many others. The Haar wavelets have the following features: (1) highly energy packing; (2) the base functions are consisted of three simple integers, 0 and -1, and 1 only. The properties are useful in speeding up the computation. So we will use the Haar wavelets and its operational matrix for demonstration throughout this paper. Let us begin by briey reviewing the Haar functions [24-25]. The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval [0,1) by

$$h_0(t) = \frac{1}{\sqrt{m}}$$

$$h_i(t) = \frac{1}{\sqrt{m}}\begin{cases} 2^{\frac{j}{2}} & \frac{k-1}{2^j} \le t \le \frac{k-\frac{1}{2}}{2^j} \\ -2^{\frac{j}{2}} & \frac{k-\frac{1}{2}}{2^j} \le t \le \frac{k}{2^j} \\ 0 & otherwise \end{cases} \qquad (4)$$

Where $i = 1,2,\ldots,m-1$, $m=2^r$ and r is a positive integer. j and k represent the integer decomposition of the index *i, i.e,* $i=2j+k-1$. Theoretically, this set of functions is complete, In the construction, $h_0(t)$ is called the scaling function and $h_1(t)$ the mother waelet. There are two basic operations involved in this set of Haar functions: (1) Translation and (2) Dilation [25]. Starting form the mother wavelet, $h_1(t)$ Compression and translation are performed to obtain $h_2(t)$ and $h_3(t)$ as shown in Fig. 1. Any function y(t) which is square integrable in the interval $0 \le t \le 1$, that is can be expanded into Haar series by

$$\int_0^1 y^2(t)\,dt < \infty \qquad (5)$$

$$y(t) = c_0h_0(t) + c_1h_1(t) + c_2h_2(t) + \cdots, \qquad (6)$$

where $c_j = \int_0^1 y(t)h_j(t)\,dt$.

$$y(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) \quad (7)$$

In usual the series expansion of (6) contains infinite terms for a general smooth function y(t). However, if y(t) is approximated as piecewise constantduring each subinterval, (6) will be terminated at finite terms, i.e.

The Continuous curve of (7) can be written into the discrete form by

$$y(t) \approx c_0 h_0(t) + c_1 h_1(t) + c_2 h_2(t) + \cdots + c_{m-1} h_{m-1}(t), \quad (8)$$

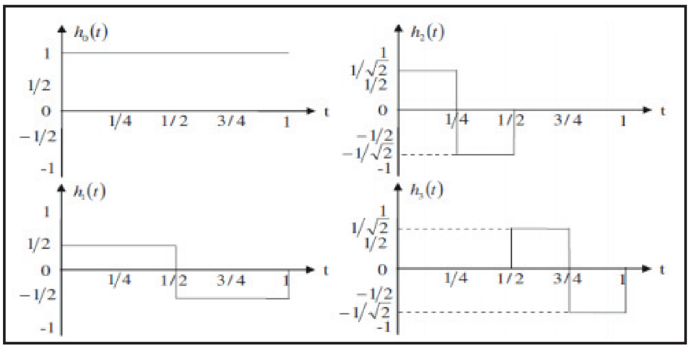

Fig. 1: Haar Wavelets Functions with m = 4.

Where M is the Dimention and usually $m=2^r$, is a positive integer. $\vec{y}^T = [y_0, y_1, ..., y_{m-1}]$ is the discrete form of the continuous function y(t), the discrete values yi are obtained by sampling the continuous curve y(t) at a space 1\m. Similarly $\vec{h}_0^T = [h_{0,0}, h_{0,1}, h_{0,2}, ..., h_{0,m-1}], \vec{h}_1^T = [h_{1,0}, h_{1,1}, h_{1,2}, ..., h_{1,m-1}], ..., \vec{h}_{m-1}^T = [h_{m-1,0}, h_{m-1,1}, h_{m-1,2}, ..., h_{m-1,m-1}]$ of the Haar wavelet bases; the discrete values are take the continuous curves $h_0(t), h_1(t), ..., h_{m-1}(t)$, respectivly

$$H = \begin{pmatrix} \vec{h}_0(t) \\ \vec{h}_1(t) \\ \cdots \\ \vec{h}_{m-1}(t) \end{pmatrix} = \begin{pmatrix} h_{0,0} & h_{0,1} & \cdots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \cdots & h_{1,m-1} \\ \cdots & . & \cdots & \cdots \\ h_{m-1,0} & h_{m-1,1} & \cdots & h_{m-1,m-1} \end{pmatrix} \quad (9)$$

Eq. 8 is than expressed $\vec{y}^T = \vec{c}^T.H$, where $\vec{c} = [c_0, c_1, \ldots, c_{m-1}]^T$ is called the coefficient vector of $\vec{y}$ and it can be calculated form $\vec{c}^T = \vec{y}^T.H^{-1}$.Similarly, a two-dimensional function y(x,t) which is square integrable in the interval 0<x<1 and 0<t<1 can be expanded into Haar series by

$$y(x,t) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{ij} h_i(x) h_j(t) \quad (10)$$

Where

$$c_{ij} = \int_0^1 y(x,t) h_i(x)\, dx \int_0^1 y(x,t) h_j(t)\, dt\,. \quad (11)$$

Thus, Eq (10) can be written into the discrete form by

$$Y(x,t) = H^T(x) \cdot C \cdot H(t) \quad (12)$$

Where

$$C = \begin{pmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,m-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,m-1} \\ \cdots & . & \cdots & \cdots \\ c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,m-1} \end{pmatrix} \quad (13)$$

is the coefficient matrix of Y, and it can be calculated by

$$C = H \cdot Y \cdot H^T \quad (14)$$

For deriving the operational matrix of Haar wavelets, we let Φ=H in Eq. (3), and obtain

$$F_H = H \cdot F_B \cdot H^T, \quad (15)$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 & 1 \\ 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} .5 & -.25 & -.0884 & -.0884 \\ .25 & 0 & -.0884 & .0884 \\ .0884 & .0884 & 0 & 0 \\ .0884 & -.0884 & 0 & 0 \end{pmatrix}.$$

For any $m=2^r$ where r is a positive integer, We can establish the corresponding operational matrix accordingly.

III. Analysis Methods

Obviously, can be extended to the case of function with to variables, the integration order α of $Y(x,t) = H^T \cdot C \cdot H(t)$ with respect to variable t yields

$$I_t^\alpha Y(x,t) = I_t^\alpha (H^T(x) \cdot C \cdot H(t)) \\ = H^T(x) \cdot C \cdot I_t^\alpha H(t) = H^T(x) \cdot C \cdot F_H^\alpha \cdot H(t) \quad (16)$$

Similarly, the fnctional integration order α of y (x,t) with respect to veriable x can be expressed as

$$I_x^\beta Y(x,t) = I_x^\beta (H^T(x) \cdot C \cdot H(t)) \\ = \left[I_x^\beta H(x)\right]^T \cdot C \cdot H(t) = F_H^\beta \cdot H^T(x) \cdot C \cdot H(t) = H^T(x) \cdot (F_H^\beta)^T \cdot C \cdot H(t) \quad (17)$$

IV. Applications and Results

Example 1: We consider the following fractional Heat Equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\pi^2} \frac{\partial^2 u(x,t)}{\partial x^2} \quad (18)$$

along with the following initial and boundary conditions

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(0,t) = \pi E_\alpha(-t^\alpha), \quad t > 0 \quad (19)$$

$$u(0,t) = \sin \pi x, \quad x > 0$$

The exact solution of (18) - (20) is $u(x,t) = \sin\pi x\, E_\alpha(-t^\alpha)$.(20)

Applying the double Laplace Transform to both sides of (5.1), we have

$$s_2^\alpha \left[L_t L_x u(x,t) - s_2^{-1} \frac{\pi}{s_1^{-1}} \right] = \frac{1}{\pi^2} \left[s_1^\alpha L_t L_x u(x,t) - \frac{\pi}{s_2} \right].$$

Hence,

$$(s_2^\alpha - \frac{s_1^2}{\pi}) L_t L_x u(x,t) = s_2^{\alpha-1} \frac{\pi}{s_1^2} - \frac{1}{\pi s_2} = \frac{\pi^2 s_2^\alpha - s_1^2}{\pi s_1^2 s_2}.$$

Thus, we have

$$L_t L_x u(x,t) = \frac{\pi}{s_1^2} . \frac{1}{s_2}.$$

Than, integration (5.6) with respaect to x two times, we get

$$\int_0^x\int_0^x u(x,t)\, dx\, dx - \pi \int_0^x\int_0^x\int_0^x dx\, dx\, dx = \frac{1}{\pi^2}\left(I_t^\alpha u(x,t) - \pi \int_0^x I_t^\alpha (1)\, dx \right)$$

$$H^T(x) \cdot (F_H^2)^T \cdot C \cdot H(t) - \pi H^T(x) \cdot (F_H^3)^T \cdot J \cdot H(t) \\ = \frac{1}{\pi^2} H^T(x) \cdot C \cdot (F_H^\alpha)^T \cdot H(t) - \frac{1}{\pi} H^T(x) \cdot (F_H^1)^T \cdot J \cdot F_H^\alpha \cdot H(t)$$

where

$$J = H \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & . & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{m \times m} \cdot H^T \,.$$

By multiplying $H^T(x)$ to the right side and H(t) to the left side of each term in (5.8), it give:

$$(F_H^2)^T \cdot C - \pi (F_H^3)^T \cdot J = \frac{1}{\pi^2} \cdot C \cdot F_H^\alpha - \frac{1}{\pi} (F_H^1)^T \cdot J \cdot F_H^\alpha$$

If the Haar wavelets with order m is used, (5.10) can be simplifed as:

$$(F_{H_m}^2)^T \cdot C_{m\times m} - \pi (F_{H_m}^3)^T \cdot J_m = \frac{1}{\pi^2} C_{m\times m} \cdot F_{H_m}^\alpha - \frac{1}{\pi} (F_{H_m}^1)^T \cdot J_m \cdot F_{H_m}^\alpha$$

 ISSN : 2348-0033 (Online) | ISSN : 2249-4944 (Print)

Solving the above equation to $m=8$ and $\alpha=\frac{1}{2}$, the matrix form of $u(x,t)$ is given by

$$\begin{bmatrix} 109.1780 & 116.1585 & 88.4466 & 75.2536 & -24.6560 & -156.6537 & -342.2363 & -586.5817 \\ 860.1673 & 822.8567 & 970.9540 & 1041.566 & 1575.925 & 2281.514 & 2371.886 & 4572.098 \\ 1077.383 & 1136.713 & 901.1135 & 789.2779 & -58.8275 & -1180.506 & -2762.631 & -4857.286 \\ 3851.943 & 3375.387 & 5266.210 & 6171.852 & 13009.15 & 22023.91 & 34617.70 & 51016.62 \\ -10971.47 & -8817.750 & -17360.79 & -21462.56 & -52400.09 & -93148.34 & -1.499\,10^5 & -2.234062^5 \\ 50170.69 & 42248.90 & 73663.42 & 88787.45 & 2.0271383^5 & 3.526208^5 & 5.6076830^5 & 8.2889910^5 \\ -1.764110^5 & -1.46910^5 & -2.638\,10^5 & -3.20310^5 & -7.45010^5 & -1.30310^6 & -2.07610^6 & -3.06710^6 \\ 6.7301010^7 & 5.641410^5 & 9.956010^5 & 1.204410^6 & 2.773410^6 & 4.834110^6 & 7.678210^6 & 1.130210^7 \end{bmatrix}$$

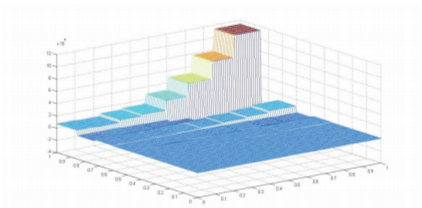

Fig. 2: The Approximate Soluation of (18-20) in the Case α = 1/2.

V. Future Plane

Application of AWCM and adaptive multilevel solver to more realistic models. Application of diffusion wavelets to partial differential equations

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