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An Approach to Pedagogy and Assessment in Mathematics

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Abstract
Mathematics requires, among other things, a clear logical thinking, an ability to identify patterns and application of known results and axioms. Solving difficult mathematical problems is a skill that requires careful planning and it comes with a lot of practice. Attempting to solve a problem without a plan would get us into endless explorations and frustration. Inadequate practice and poor understanding of the problem solving process leads to fear of mathematics among students. This paper suggests an approach to teaching problem solving skills in mathematics with an emphasis on learning one thing at a time. While the approach can be adopted in a typical classroom setting without the aid of computers, use of a software tool that is programmed to pose the relevant questions would help in saving the time for teachers and offer flexibility to students in choosing the right time for their practice.

Keywords
Problem Solving, Assessment, Pedagogy, Mathematical thinking, Sandbox approach, Meta-problems.

I. Introduction

Mathematics is fun for some students whereas it is a difficult subject for others. Researchers have studied the math learning disabilities among students and have identified fear as one of the main causes. Slow learners are also found to be deficient in memory, attention, active learning and meta-cognitive thinking. While the best cure for math fear seems to be 'success', we need to find ways of helping students achieve it. It is a generally known fact that small successes can give us the confidence to attempt more difficult tasks. What exactly is small success? This paper attempts to explore this question and devise methods with the objective of dispelling math fear and create a love for the subject.

II. Mathematical Problem Solving

Several researchers [1]-[11] have studied the mathematical problem solving process with the objective of helping students develop their math skills. George Polya, an eminent mathematician identified 4 broad steps that form part of the problem solving process and described them in his famous book titled 'How to Solve It' [1]. These broad steps are,

1. Understand the problem
2. Devise a plan
3. Execute the plan
4. Check the result

These are often neglected as being obvious. He has elaborated each of these steps in his book with examples, identified several heuristics that were devised by mathematicians and stressed the need for application of strategies and heuristics. He advises "If you can't solve a problem, then there is an easier problem you can solve: find it". And, "If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?" he asks.

Alan Schoenfeld, a math educationist who is the author of several books including 'Mathematical Thinking and Problem Solving' [2] and 'How We Think' [3], stressed the importance of metacognition, in the sense of 'self-regulation during problem solving'.

III. Proposed approach

While the proposed approach utilizes the main elements of the early work of George Polya for developing the math problem solving skills, there are some key elements when it comes to its execution. They have been identified and presented here. These key elements are especially relevant today when IT can provide us with software tools that are extremely user friendly and always available. The key elements can be summarized as follows:

A. Focus on one aspect at a time
B. Use graphical tools
C. Use sandbox approach
D. Consider meta-problems
E. Gamify

A. Focus

To be skilled in math problem solving one needs to develop a thorough understanding of every aspect of the problem solving process. This requires practice not just on different problems, but on each of these aspects. Focus facilitates small successes, which in turn reinforce a student's confidence in solving complex problems. Focus is illustrated in the next section with several examples.

B. Graphical Tools

A picture is worth a thousand words. A picture helps in quick understanding of complex ideas. It puts less stress on the student if he is expected to provide an answer to a question using a graphical user interface by just selecting the right options or joining the right graphical elements on the screen.

C. Sandbox Approach

In a typical classroom environment, students are given a problem, that is either simple or complex and are expected to solve it in a certain timeframe. In the sandbox approach, the students would be given a small set of premises and are asked to identify correct inferences from the given set of options. Once he completes it, more premises would be added or an existing premise modified or removed and they would be asked to identify correct inferences as a result of it. In this approach the students do not solve 'a problem' in the traditional sense of the word. They are given a sequence of small exercises each resembling a step in a solution to a problem, yet do not contribute towards solving a typical mathematical problem. This approach has several benefits. It not only helps a student focus on one small problem at a time but it also enables us to test the student on a variety of concepts and strategies and find his weaknesses at a very fine level of granularity. Offering more questions on his weak areas in a focused manner would help him strengthen himself in those areas. This approach avoids having to solve many lengthier problems covering only a few concepts. In a short time and with less effort, a student can cover many different concepts and strategies that he has to learn to become a good problem solver.

D. Meta-problems

These are problems about problems. Students may be given a problem and asked to identify its relationship with another problem. Alternatively, students may be given a set of problems and asked to identify the odd man out, in terms of strategies to be adopted or the underlying patterns in them.

E. Gamification

Gamification is the application of game design elements to non-game contexts. It is different from game based learning in the sense that the user does not have to play a game on the computer to complete a given task. Gamification leverages people's natural desire for socializing, competition, achievement and status. They use different strategies like conducting a competition among players to engage them in a task or rewards for players who accomplish the desired tasks. Rewards could be in the form of points, levels and achievement badges. It can be easily seen that these approaches can very well be applied to our problem solving scenario.

IV. Approach in Detail

In this section we will see several types of questions in mathematics that exhibit the above features. The exact questions from mathematics are not shown here. Actual problem statement will replace the words shown in square brackets.
Q1: You are given the following problem statement:
[Problem statement]
From the choices given below, pick the ones that represent a right understanding of the problem (See fig. 1).

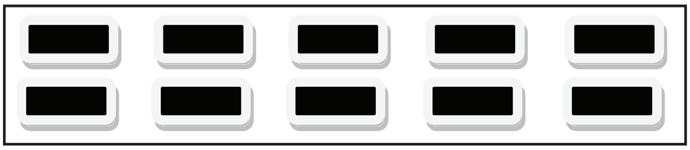

Fig.1: Understanding the Problem

The student is expected to select the options that can be inferred from the problem statement. In this problem, 10 choices were given. It can be more than 10 or even less than 10. The layout of the question and the options can be changed to suit the length of the text.
Asking many such questions on the topic of 'Understanding' helps students focus on understanding of the problem. He has to just focus on it rather than worry about further steps in the solution.

Q2: Given the Statement, [Statement]
Which of the following represents a correct reformulation of the statement (See fig. 2).

Fig. 2: Reformulation of the Statement

The statement here, can be any kind of mathematical statement or assertion; it need not be a typical problem statement.

Some more question types are shown here. Every question statement will be followed by multiple choices from which the student has to choose one or more options. To save space, the multiple choices are not shown against the questions.

Q3: Identify the constraints in the given problem.
Q4: Identify the correct reformulation of the given constraint.
Q5: What patterns do you see in the given premises that are of relevance to the solution?
Q6: You are given a mathematical expression. In addition, some standard forms of expressions are shown below. Pick the standard forms to which the given expression can be converted.
Q7: You are given a problem and a partial sequence of steps for its solution. Find out what you can infer at this stage.
Q8: How do you infer that? Specifically, which prior steps or axioms or prior results enabled you to infer that?
Q9: Given a problem in geometry, if you want to construct a line (or a tangent), where would you draw that line?
Q10: For the given expression (or a geometric entity) is there any specific property that should be considered to solve the given problem?
What are the conditions / criteria to consider?
Does it satisfy the condition?
Is the condition necessary or sufficient?
Q11: Given the problem statement, can you identify a sub goal to this problem from the below list?
Q12: What all can you say about this expression / about this variable / set (as its property)?
Q13: Given a problem, which of the following observations / operations / constructions / plans or steps appear(s) relevant to solving the problem?
Q14: Which of the following comparisons appears more relevant to the problem at hand?
Q15: Given a problem and some sequence of steps to solving it, strike out the steps that are irrelevant to solving the problem.
Q16: Given a problem, some solution sequences are given. Find out which of them will help in solving the problem.
Q17: Given an expression, find the relevant upper bounding expression from the given set of options. (More generally, it can be any relation)
Q18: Given an expression, find the expression from the set of options that is greater than the given expression. (More generally, it can be any relation and expression can again be replaced by any mathematical entity)
Q19: You are given below a set of patterns or statements on the left hand side and some problems on the right hand side. Match the left hand side entry that would be of relevance to solving the problem(s) on the right hand side (See fig. 3).

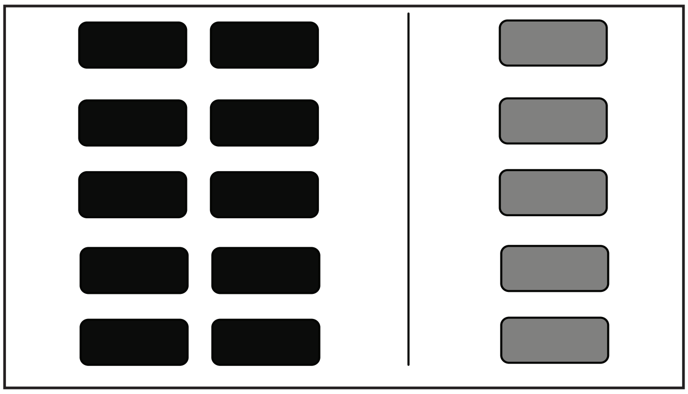

Fig. 3: Match the Problems With Patterns

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Q20: You are given a set of premises and an inference. Out of the given premises, pick the complete set of premises that are needed to establish the inference shown.

Q21: You are given a problem statement, some initial steps and some final steps. From the given set of options, find the correct intermediate steps in the correct order.

Q22: A logical sequence of steps is given. From the given options, find the correct inference from these steps.

Q23: In the given problem, if the given premise is changed this way [description of change here], which of the following would be applicable, for solving the problem.

Q24: For the given technique, select the problems where it is not applicable.

Q25: Match the given set of problems (on the left) with the techniques listed on the right that can be applied to solving them.

Q26: Match the known results given on the left hand side to the problems given on the right hand side that they help in solving.

Q27: Find out all logical sequences of inferences from the given set of propositions which are not necessarily part of any specific math problem. Join the premises to inferences by arrows (See fig. 4). (The actual diagram may be larger than what is shown here).

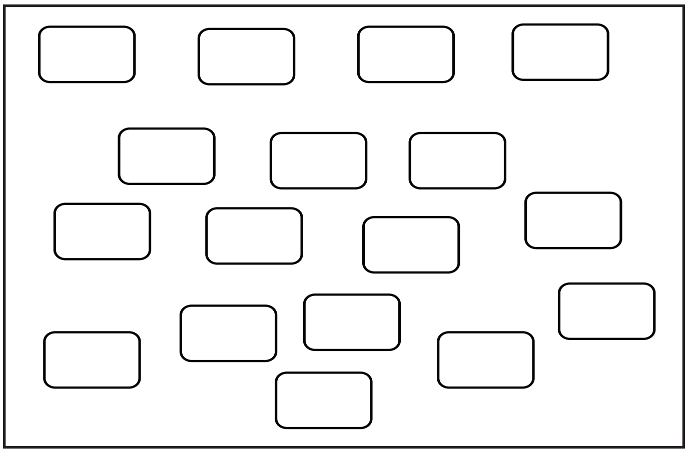

Fig. 4: Logical Sequences

The student is expected to identify the inferences and for any identified inference he should identify the set of premises which together lead to that inference. Once he identifies them, he should join all those premises to that inference. A premise may lead to more than one inference. And an inference may come from more than one premise in more than one way. Answer to this problem would be like this (See fig. 5):

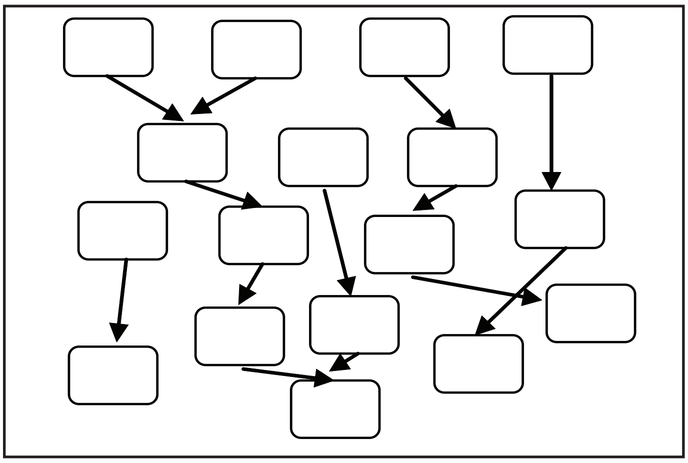

Fig. 5: A Typical Set of Logical Sequences

The arrows here denote 'implies' relationship. The square boxes are assertions or intermediate or final inferences.

Q28: For the problem [problem statement], a proof is given below on the Left hand side. From the list of axioms and known results shown on the right hand side and also from the previous steps, find out which of them have been used in which steps of the solution (See fig. 6).

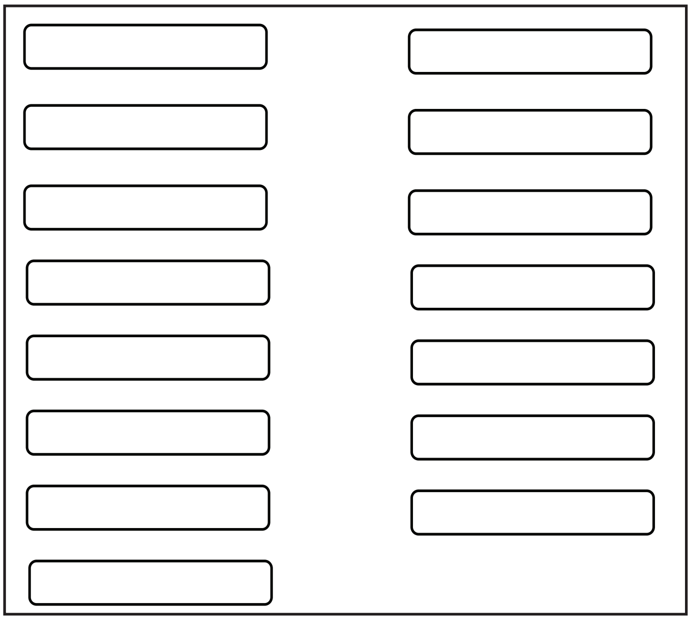

Fig. 6: Fill the Proof With Logical Links

The student is expected to analyze every step starting from the top, identify the premises and inferences and join them as in the previous example, i.e. Q27. The difference between Q27 and Q28 is that Q27 does not represent a single mathematical problem in the traditional sense whereas Q28 is the case of solving a single mathematical problem. This question type may be further enhanced by asking the student to input some numbers or algebraic expressions or joining some geometric entities, wherever required.

A solution expected from the student may look like this (See fig. 7):

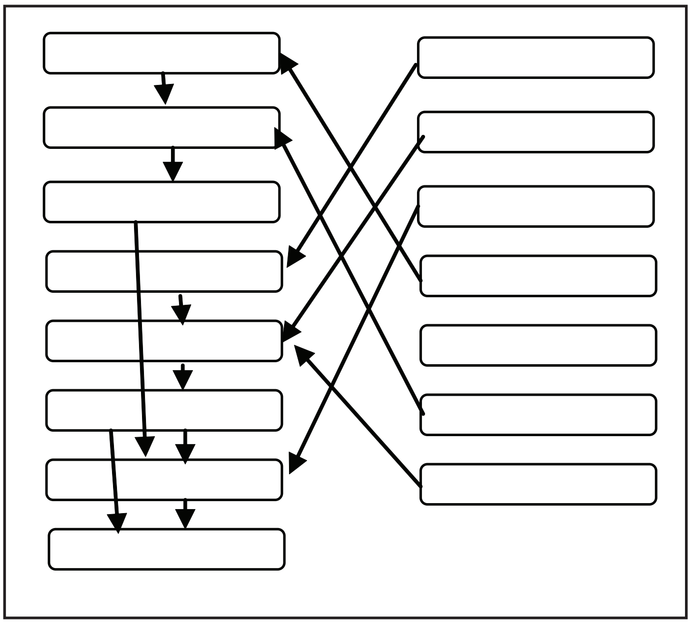

Fig. 7: Answer to filling the proof with links

Q29: For the given problem [problem statement], find out which of the following may be sub-problems.
Q30: For this given result or theorem, find out which of the following can be a corollary.
Q31: Given these expressions [expression list], what all patterns do you see in these expressions.
Q32: For the given statement, find out which of the following is the correct mathematical representation.
Q33: For the given problem [problem statement], Construct the goal – sub goal hierarchy by choosing the right options from the following sequence for sub goals.
Q34: Given the problem statement [problem statement], pick the right sequence of steps to solve the problem. Completion of every step will be a 2 stage process. First, the student will have to select certain options to demonstrate his understanding of the current state of the solution. Next he will have to choose what he will do based on that understanding. These 2 stages will apply to every step till the problem is completely solved.

V. Discussion

There are 34 question types listed in the previous section. It can be seen that these question types exhibit one or more of the 5 key elements of our approach, listed in Section III. Many of them starting from Q1 exhibit focusing on one aspect of the problem. Q1 for instance focusses on understanding phase of the solution process. Q6, Q10 and Q11 are about the right heuristics that can be applied. Q27 is about the sandbox approach. Many of the problems show the use of graphical elements like joining two boxes. This reduces effort for the student and develops his conceptual understanding. Q23 to Q26 are meta-problems because they are questions about the problems themselves.
By focusing on one aspect of the problem, the student gets lot more practice on that aspect in a shorter time than if he solves entire problems given to him. Focus helps in achieving a thorough understanding of something that is focused upon.
With user-friendly graphical tools, student solves the problems with much lesser effort, as he will have to just select some entities, join some entities or steps of the solution or enter some numbers.
Sandbox approach gives lot of practice on very small inference tasks. It is a network of inferences that can be extended to any size. This again lets a student get exposure on different types of strategies and heuristics that he has to learn, without having to solve many lengthy problems.
Gamification can be incorporated by giving points to each and every small tasks. As a student solves more and more problems he can be assigned higher levels and appropriate badges based on the skills he has exhibited. By adding timer control, competitions can be held on such interesting tasks and winners can be given awards.

VI. Future Work

The author intends to have these ideas experimented in schools, especially for 8th grade and above. It would be nice to have a software tool that implements these ideas. It requires effort towards creating a large set of questions of different complexities, identifying the premises and inferences, identifying the strategies and heuristics that are applied at different steps, grouping the problems depending on their similarities and tagging the steps with the type of operation or idea that is required to complete that step.

VII. Conclusion

We have described an approach to mathematical problem solving with a stress on 5 key elements namely, focusing on one aspect at a time, using graphical tools, using sandbox approach, considering meta-problems and incorporating gamification. It is hoped that adopting this approach can help students overcome fear and develop love for the subject.

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