Representations by Quadratic Forms: A Review of Significant Developments

[1]Ms. Chetna, [2]Dr. Hardeep Singh

[1]Assistant Professor, Dept. of Mathematics, M.M. Modi College, Patiala, Punjab, India

[2]Professor and Head, Dept. of Mathematics, Government Mahindra College, Patiala, Punjab, India

Abstract

This paper examines and highlights the developments in the area of the theory of representations by the quadratic forms over the field of integers. The main aim of this paper is to take stock of the evolution and subsequent contributions in the area of integral representations by quadratic forms. From the review of existing literature, it is quite evident that the representation theory of quadratic forms has an extensive history. It started in the 17th century with Fermat's statements about the prime numbers that are represented in the form $x^2 + y^2$. Then, with the classical contributions of Euler for binary quadratic forms, Lagrange's four square theorem, and Legendre's three square theorem, and after that with the significant work by Ramanujan on the representation by the sum of four squares and Dickson's results about positive ternary quadratic forms – this knowledge area has become a specialized domain in itself. Recently, the work of Bambah, Dumir, Hans-gill, Conway and Schneeberger, Bhargava and Hanke has also significantly enriched this area.

Mathematics Subject Classification (2015)

11R04 , 11E88.

Keywords

Rings, Integers, Quadratic Forms, Representation

I. Introduction

Quadratic forms are homogeneous quadratic polynomials in n variables. If it contains only one variable then it is called unary and in the cases of two, three and four variables they are called binary, ternary and quaternary quadratic forms. The methods used in the study of quadratic forms depend to a large extent on the nature of the coefficients, which may be real, or complex numbers, rational numbers, or integers. The theory of quadratic forms as a branch of number theory deals with quadratic Diophantine equations initially over the field of integers. Then two main problems arise one deal with when two quadratic modules are equivalent and other deal with which integers are represented by the given quadratic forms. The main aim of this paper is to take stock of the evolution and subsequent developments in the area of integral representations by quadratic form. From the review of existing literature it is quite evident that the representation theory of quadratic forms has an extensive history. It started from the 17th century with Fermat's statements about the prime numbers that are represented in the form $x^2 + y^2$. From this it is clear that not every prime number can be written as a sum of two squares. Any prime which is not of the form 4k + 3 can be written as a sum of two squares. In this context Fermat gave the following result that if p is prime and either p is congruent to 1(mod 4) or p = 2 then $p = x^2 + y^2$ over Z. In the subsequent century, Euler proved these and gave some comparable statements about other simple binary quadratics. Though these proofs given by him had some loopholes, they contributed to a great extent to the theory of quadratic forms. After that in 1770 Lagrange established the theory of general quadratic forms by proving his well-known Four Squares Theorem, which today is being known by the statement that every positive integer can be expressed in the form $x^2+ y^2+ z^2 + t^2$. In 18th century, a notably deeper statement given by Legendre in his Three Squares Theorem (1798) stated that every positive integer can be expressed as a sum of three squares if and only if it is not of the form 4K (8m+7). Legendre also produced a broad-spectrum theory of binary quadratics.

Quadratic Form

A function $f : R_n \rightarrow R$ of the form $f(x) = X^T AX = \sum_{i,j=1}^{n} A_{i,j}\ x_i x_j$ is called a quadratic form, in which each term is a monomial of degree 2. In a quadratic form we may additionally assume $A = A^T$ since $X^T AX = X^T\ ((A + A^T\ )/2)X$ Since F is a field, $x_i x_j = x_j x_i$ for any *i, j,* we can make the coefficients symmetric by rewriting *f* as $f(X_1,\bullet\bullet\bullet, X_n) = \sum\sum_{i,j=1}^{n} \frac{1}{2}(a_{ij} + a_{ji})X_i X_j$ Now f determines uniquely a symmetric matrix M_f where $M_f = 1/2(\ a_{ij}+a_{ji})$.

Uniqueness

If $X^T AX = X^T BX$ for all $X \in R_n$ and $A= A^T$, $B = B^T$, then A = B. An integral binary quadratic form is a polynomial f(x,y) of the form $f(x,y) = ax^2 +bxy +cy^2$, where $a,b,c \in Z$. We shall generally reduce this formula by writing f = [a,b,c]. Let f and g be quadratic forms, we say that f is equivalent to g (or $f \approx g$) if there exists an invertible matrix $A \in GL_n (F)$ such that f(X) = g(AX). Notice that $f(X) = g(AX) = (AX)^T M_g (AX) = X^T (A^T M_g A) X$ this implies that $M_f = A^T M_g A$. Thus, equivalence of forms can be regarded as congruence of the associated symmetric matrices and it is an equivalence relation.

Quadratic Map

We can define the quadratic map Q_f defined by f to be $Q_f : F_n \longrightarrow F$ such that $Q_f (x) = x^t M_f x$, for any x F_n viewed as a column vector. In relation with the equivalence of forms, $f \approx g$ amounts to the existence of a linear automorphism C of F_n such that $Q_f (x) = Q_g (Ax)$ for every column tuple x. That means the quadratic map Q_f determines uniquely the quadratic form f. We also have the property that$Q_f (ax) = a^2 Q_f (x)$ for any $a \in F$.

For the standard results we have referred to work of Burton [3] and Niven[18]. Between 1884 and 1890 Minkowski developed the fundamentals of a general theory of quadratic forms over the field rational and rational integers. A remarkable work of the very young Minkowski was the paper on Foundations of a theory of quadratic forms with rational coefficients. The paper written by him mainly deals with the local classification of integral quadratic forms. In those times, the rational theory still was a consequence of the integral theory. Later on his paper on the conditions under which two quadratic forms over rational can be transformed into each other basically deals with the main theorem over Q. He also associates a system of invariants equal to ±1, one for each prime. He shows that these invariants together with the discriminant determine the equivalence class over rational. This result shows that for equivalence of quadratic form the local-global principle is weak but theorem is not yet used. For these results we referred to the work of Timothy[22] and Scharlau[21]. Dickson [4] in his paper on integers represented by positive ternary forms proved the theorem that $x^2+y^2+3z^2$ represent exclusively all positive integers not of the form $9^k(9n+6)$ and in 1927 he confirmed that (1, 2, 5, 5) does not belong to the list given by ramanujan as it does not represent 15. Dumir [6-7] worked on Inequalities of Non-homogenous ternary quadratic Forms and mentioned that if Q (x,y,z) be an indefinite ternary quadratic form of determinant $D<0$ and $t\geq 0$ be any given real number then there is existence of a function such that for given reals say a,b,c we can find the integers. Dumir [8] worked on Positive values of inhomogeneous quadratic forms 1. He proved that if there exists a real indefinite quadratic form in n variables with signature (r, n-r), $0< r < n$ and determinant $D \neq 0$ then there exist some constant Γ such that $\Gamma_{r,n-r}$ denote the greatest lower bound of all such constants Γ. For form 1 he evaluated Γ for signatures r-s = 1,2, or 3. Bambah, Dumir, Hans-gill [1] modified the conjecture of Jackson on non-homogenous quadratic forms. They proved that the conjecture is true for indefinite quadratic forms of signature 0,+1,-1or $\pm$ 2,-2. For forms 2, Bambah, Dumir and Hans-Gill [2] evaluated Γ for $r\geq 2$ and showed that this value is isolated and this result is used to prove that $\Gamma_{r,r+1} = 2^{2r}$ for all $r \geq 3$. Dumir and Hans-Gill [10] carried the work on positive values of non-homogeneous quaternary quadratic forms of type (1,3) and showed that $\Gamma_{1,3} = 16$. Dumir [9] examined the positive values of non-homogeneous quaternary quadratic forms 1 and 2. Hans-gill and Raka [14-15] worked on positive values of 5-ary quadratic forms of type (3,2) and showed that the only value for this type is 16 . Further, Hans-gill and Raka [14-15] continued their work on quinary quadratic forms of type (4,1) and found the minimum value 8 for (4,1). Raka and Rani [20] proved that for all non-homogeneous indefinite ternary quadratic forms, whether zero or non zero of type (2, 1) the minimum value is 8/3. Dumir and Sehmi [11,13] in their paper on positive values of non homogeneous indefinite quadratic forms of type (3,2) showed that second minimum for this type is 8 and also continued their work for type (1,4). Dumir and Sehmi [12] in their paper used the results of their previous paper to determine the second minimum values of non homogeneous indefinite quadratic forms of signature 1 for $n \geq 5$ variables. Further, Raka [19] found the minimum values of non-homogeneous quadratic forms of signature ± 1. Duke [5] discussed some old problems and new results about the quadratic forms. In his paper Duke discussed the results given by Ramanujan. In this paper he explained that Ramanujan considered the problem of finding all integers $0 \leq a \leq b \leq c \leq d$ for which every positive integer is represented in the form $ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2$. A straightforward and entertaining case by case investigation shows that in order for 1,2,3,5 to be represented, the first three terms (a; b; c) must be (1,1,1), (1,1,2), (1,1,3), (1,2,2), (1,2,3), (1,2,4), or (1,2,5). None of the associated ternary forms $ax_1^2 + bx_2^2 + cx_3^2$ represents all numbers, the least exemptions being, respectively 7, 14, 6, 7, 10, 14, and 10. This leaves 55 possible quaternary forms, and, based on straightforward rules for the integers represented by the above ternaries which Ramanujan discovered empirically. He concluded that these 55 forms in fact do represent all numbers [5]. Then work of ramanujan was generalized to other quadratic forms. Willerding [23] determined all classes of positive quaternary quadratic forms which represent all positive integers and after extending the work of Ramanujan stated that there are exactly 178 classes of universal classic positive quaternary quadratic forms. Further after taking into consideration the list of Ramanujan, Conway and Schneeberger conclude the result that if any positive quadratic form over Z which represents the values 1, 2, 3, 5, 6, 7 10, 14, 15 then it represents all positive integers. Then Bhargava has generalized the Fifteen theorem and stated that if we consider any set A of non-negative integers then there must exist a unique subset B of A such that a classic definite quadratic form represents A if and only if it represents B. Later 290-theorem was proved by Bhargava and Hanke which removes the supposition classic from the hypothesis of the Fifteen Theorem which states that if any quadratic form represents 290 then it represents every non-negative integer [16-17]. From the above review of literature we find that the representation theory or solutions given by the quadratic form has important role in the field of number theory and we can also conclude that there is an association between representation of elements by the quadratic form and solubility of the quadratic form over the field of integers.

References

[1] Bambah, Dumir V. C., Hans-Gill,"On conjecture of Jackson on non-homogenous", J, of Number Theory, 16, 1983, pp. 403-419.

[2] Bambah, Dumir V. C., Hans-Gill,"Positive values of non-homogenous indefinite quadratic form 2", Journal of Number Theory, 18, 1984, pp. 313-341.

[3] Burton David M.,"Elementary Number Theory", 6th Ed., Tata McGraw-Hill, 2010.

[4] Dickson L.E.,"Integers represented by positive ternary quadratic forms", Bull. Amer. Math. Soc. 33, 1927, pp. 63-70.

[5] Duke William,"Some old problems and new results about quadratic forms", Notices Amer. Math. Soc. 44 no. 2, 1997, pp. 190–196.

[6] Dumir V. C.,"Asymmetric inequality for non-homogenous ternary quadratic forms", proc. Cambridge. Soc. 63, 1968, pp. 291-303.

[7] Dumir V. C.,"Asymmetric inequality for non-homogenous ternary quadratic forms", J. of Number Theory, 1, 1969, pp. 326-345.

[8] Dumir V. C.,"Positive values of inhomogeneous quadratic forms 1", J Australia math. soc. 8, 1968, pp. 87-101.

[9] Dumir V. C.,"Positive values of in homogeneous quaternary quadratic forms 1 and 2", J. of Aust. Math, 8, 1968, pp. 287-303.

[10] Dumir V. C., Hans-Gill,"Positive values of non-homogeneous quaternary quadratic forms of type (1,3)", Indian J. Pure and Applied Math , 12, (1981), pp. 814-825.

[11] Dumir V. C., Sehmi Ranjeet,"Positive values of non-homogeneous indefinite quadratic forms of type (3,2)", Journal of Pure and Applied Mathematics, 24(7), pp. 827-853, 1992.

[12] Dumir V. C. and Sehmi Ranjeet, Positive Values of Non Homogeneous Indefinite Quadratic Forms of Signature 1, Indian J. pure appl. Math., 23(7), (1992), 855-864.

[13] Dumir V. C. and Sehmi Ranjeet, Positive values of non-homogeneous indefinite quadratic forms of type (1, 4), Proceedings of the Indian Academy of Sciences - Mathematical Sciences, Volume 104, Issue 3, (1994), 557-579.

[14] Hans-Gill and Raka Madhu, Positive values of in homogeneous 5-ary quadratic form of type (3,2) , J. Austrial Math , 29, (1980), 439-453.

[15] Hans-Gill and Raka Madhu, Positive values of inhomogeneous quinary of (4,1) , J. of Math , (1981), 175-188.

[16] Manjul Bhargava, Higher composition laws I: A new view on

ISSN: 2348-0033 (Online) | ISSN : 2249-4944 (Print)

Gauss composition and quadratic generalizations, Annals of Mathematics, 159 (2004), 217–250. Accessed online: http://annals.math.princeton.edu/wp-content/uploads/annals-v159-n1-p03.pdf
[17] Manjul Bhargava and Jonathan Hanke, Universal quadratic forms and the 290- Theorem, [Onlne] Available: http://math.stanford.edu/~vakil/files/290-Theorem-preprint.pdf
[18] Niven, Zuckerman, Montgomery,"An Introduction to the Theory of Numbers", 5th Ed., Wiley India Pvt. Ltd., 2010.
[19] Raka Madhu,"The inhomogeneous minimum of quadratic forms of signature ± 1", Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 89, Issue 02, pp. 225-235, 1981.
[20] Raka Madhu, Rani Urmila,"Positive values of inhomogeneous indefinite ternary quadratic forms of type (2, 1)", Hokkaido Math. J. Vol. 25, No. 2, pp. 215-230, 1996.
[21] Scharlau, Winfried,"On the history of the algebraic theory of quadratic forms", In Quadratic forms and their applications (Dublin), Vol. 272 of Contemp. Math., (1999), 229.
[22] Timothy O'Meara,"Introduction to Quadratic Forms. Springer-Verlag, 1963. Reprinted in the Classics of Mathematics Series, Springer-Verlag, 2000
[23] Willerding F. Margaret, Determination of all classes of positive quaternary quadratic forms which represent all positive integers, Dissertation, St. Louis University, 1947.