Subordination Properties for Certain Subclass of Uniformly β-Starlike and β-Convex Functions by Using Convolution

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Abstract
In this paper we derive some subordination properties for certain subclasses of uniformly β-starlike and β-convex functions by using convolution.

Keywords
Convolution, Subordinance factor sequence, Subordinance principle, Analytic, Univalent, Uniformly.

I. Introduction

Let C(k) denote the class of all univalent and analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} \alpha_k z^k \quad (1)$$

in the open unit disc U = {z: |z|<1} Let f ∈ C (k) given by (1) and g ∈ C (k) given by

$$g(z) = z + \sum_{k=2}^{\infty} \beta_k z^k \quad (2)$$

We define the convolution product (or Hadamard) of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} \alpha_k \beta_k z^k = (g * f)(z);\ (z \in \mathcal{U}), \quad (3)$$

We also denote the class of functions f(z) ∈ C (k) that are convex in U as K.

We denote subclasses of C(k) by $S_p(\alpha,\beta)$ and K (α,β) consisting of all functions which are uniformly β-starlike and uniformly β-convex respectively (0≤α<1 ,β≥0 ;z∈U) Thus,

$$S_p(\alpha,\beta) = \left\{ f \in C(k): Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta \left|\frac{zf'(z)}{f(z)} - 1\right| \right\}, \quad (4)$$

and

$$K(\alpha,\beta) = \left\{ f \in C(k): Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \beta \left|\frac{zf''(z)}{f'(z)}\right| \right\}, \quad (5)$$

The classes Sp(α,β) and K (α,β) were introduced by Goodman ([5],[6]) , Ronning ([9],[10]). It follows from (4) and (5) that

$f(z) \in K(\alpha,\beta) \Leftrightarrow zf'(z) \in S_p(\alpha,\beta)$ (6)

Definition 1 [12]

(Subordination Principle). For two functions f and g analytic in U. We say that the function f(z) is subordinate to g(z) in U, and written symbolically as f(z)≺g(z), if there exists a Schwarz function w(z), which (by definition) is analytic in U with w(0) = 0 and |w(z)|<1, such that f(z) = g(w(z)) (z∈U). Indeed it is known that

f(z)≺g(z)⇔f(0)= g(0) and f(U)⊂g(U)

In particular, if the function g(z) is univalent in U, we have the following equivalence

f(z)≺g(z)⇔f(0)= g(0) and f(U)⊂g(U).

Definition 2 [14]

(Subordination Factor Sequence). A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be subordinating factor sequence if, whenever f(z) of the form (1) is analytic, univalent and convex in U, we have the subordination given by

$$\sum_{k=1}^{\infty} \alpha_k c_k z^k \prec f(z)\ (z \in \mathcal{U};\ \alpha_1 = 1), \quad (7)$$

For A,B fixed -1 ≤ B < A ≤1 and 0 ≤ γ ≤ 1, β ≥ 0 We define the subclass S_γ (f, g; A, B; β) of C(k) consisting of functions f of the form (1) and functions g of the form (2) with $\beta_k \geq 0$ as follows

$$\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} - \beta\left|\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} - 1\right| \prec \frac{1+Az}{1+Bz} \quad (8)$$

Where

$$zF'_\gamma(f,g)(z) = z(f*g)'(z) + \gamma z^2 (f*g)''(z)$$

$$F_\gamma(f,g)(z) = (1-\gamma)(f*g)(z) + \gamma z (f*g)'(z)$$

From (8) and definition of subordination we obtain

$$\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} - \beta\left|\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} - 1\right| = \frac{1+Aw(z)}{1+Bw(z)}$$

And hence

$$\left|\frac{\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} - \beta\left|\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} - 1\right| - 1}{B\left[\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} - \beta\left|\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} - 1\right|\right] - A}\right| < 1 \quad (9)$$

II. Main Result

To prove main result we need the following lemmas.

Lemma 1[14]

The sequence $\{c_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

$$Re\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0\ (z \in \mathcal{U}) \quad (10)$$

Now, in the following lemma we will prove sufficient condition for functions in the class S_γ (f, g; A, B; β)

Lemma 2: Let f(z)∈ S_γ (f,g;A,B;β) if

$$\sum_{k=2}^{\infty} [(k + \beta(k-1))(1-B) + (A-1)]\,[1 + \gamma(K-1)]|\alpha_K|\beta_K \leq A - B \quad (11)$$

Proof: Let equation (11) hold true then we have

$$\left|zF'_\gamma(f,g)(z) - \beta e^{i\theta}\left|zF'_\gamma(f,g)(z) - F_\gamma(f,g)(z)\right| - F_\gamma(f,g)(z)\right| - \left|AF_\gamma(f,g)(z) - B\left[zF'_\gamma(f,g)(z) - \beta e^{i\theta}\left|zF'_\gamma(f,g)(z) - F_\gamma(f,g)(z)\right|\right]\right|$$

$$= \left|\sum_{k=2}^{\infty}(k-1)[1+\gamma(k-1)]\alpha_k\beta_k z^k - \beta e^{i\theta}\left|\sum_{k=2}^{\infty}(k-1)[1+\gamma(k-1)]\alpha_k\beta_k z^k\right|\right| - \left|Az + A\sum_{k=2}^{\infty}[1+\gamma(k-1)]\alpha_k\beta_k z^k - B\left[z + \sum_{k=2}^{\infty} k[1+\gamma(k-1)]\alpha_k\beta_k z^k - \beta e^{i\theta}\left|\sum_{k=2}^{\infty}(k-1)[1+\gamma(k-1)]\alpha_k\beta_k z^k\right|\right]\right|$$

 ISSN : 2348-0033 (Online) | ISSN : 2249-4944 (Print)

$$\leq (1+\beta)\sum_{k=2}^{\infty}(k-1)[1+\gamma(k-1)]|\alpha_k|\beta_k|z|^k-(A-B)|z| + \sum_{k=2}^{\infty}(A-Bk)[1+\gamma(k-1)]|\alpha_k|\beta_k|z|^k + |B|\beta\sum_{k=2}^{\infty}(k-1)[1+\gamma(k-1)]|\alpha_k|\beta_k|z|^k$$

$$\leq (1+\beta)\sum_{k=2}^{\infty}(k-1)[1+\gamma(k-1)]|\alpha_k|\beta_k-(A-B) + \sum_{k=2}^{\infty}(A-Bk)[1+\gamma(k-1)]|\alpha_k|\beta_k + |B|\beta\sum_{k=2}^{\infty}(k-1)[1+\gamma(k-1)]|\alpha_k|\beta_k$$

$$\leq \big(1+\beta(1-B)\big)\sum^{\infty}(k-1)[1+\gamma(k-1)]|\alpha_k|\beta_k \frac{[(2+\beta)(1-B)+(A-1)]\Gamma_2(\alpha_1)}{2\{[(2+\beta)(1-B)+(A-1)]\Gamma_2(\alpha_1)+(A-B)\}}]|\alpha_k|\beta_k \sum_{k=2} - (A-B)$$

$$\leq \sum_{k=2}^{\infty}\big[\big(k+\beta(k-1)\big)(1-B)+(A-1)\big][1+\gamma(k-1)]|\alpha_k|\beta_k - (A-B)$$

$$<0$$

Hence the proof of Lemma 2 is completed ▪

Let S^*_γ (f,g; A,B; β) denote the class of f(z)∈ C(k) whose coefficients satisfy the condition (11). We note that S^*_γ (f,g; A,B; β)⊂ S_γ (f,g;A,B;β) e employing earlier used technique by Attiya [1] and Srivastava and Attiya [4] we prove,

Theorem 1: Let f(z) ∈ S_γ* (f,g; A,B; β) then

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}(f*h)(z) \prec h(z), (z\in\mathcal{U}) \quad (12)$$

For every function h(z) in K, and

$$Re\big(f(z)\big) > -\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)}{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}, (z\in\mathcal{U}) \quad (13)$$

The constant factor

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}$$

in the subordination result cannot be replaced by a larger one.

Proof: Let $f(z) \in S^*_\gamma$ (f, g; A, B; β) and let $h(z)=z+\sum_{k=2}^{\infty}c_kz^k\in K$. Then we have

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}(f*h)(z) =$$

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}\left(z+\sum_{k=2}^{\infty}c_kz^k\right). \quad (14)$$

Then by definition 2 the subordination result (12) will hold true if the sequence

$$\left\{\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}\alpha_k\right\}_{k=1}^{\infty} \quad (15)$$

is a subordinating factor sequence, with α_k = 1. In the view of Lemma 1, this is equivalent to the following inequality:

$$Re\left\{1+\sum_{k=1}^{\infty}\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}\alpha_kz^k\right\}>0, (z\in\mathcal{U}) \quad (16)$$

Now since

$$\varphi(k) = \big[\big(k+\beta(k-1)\big)(1-B)+(A-1)\big][1+\gamma(k-1)]|\alpha_k|\beta_k$$

Is and increasing function of k (k ≥2), we have

$$Re\left\{1+\sum_{k=1}^{\infty}\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}\alpha_kz^k\right\}$$

$$= Re\left\{1+\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}z + \frac{1}{\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}\sum_{k=2}^{\infty}[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2\alpha_kz^k\right\}$$

$$\geq 1-\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}r - \frac{1}{\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}\sum_{k=2}^{\infty}\big[\big(k+\beta(k-1)\big)(1-B)+(A-1)\big][1+\gamma(k-1)]|\alpha_k|\beta_kr^k$$

$$> 1-\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}r - \frac{(A-B)}{\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}r$$

$$=1-r>0 \quad (|z|=r<1),$$

This proves inequality (12). Inequality (13) can be proved by taking the convex function

$$h(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$$

to prove the sharpness of the constant factor

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}x$$

, we consider the function f_0 (z) ∈ S_γ* (f,g; A,B; β) given by

$$f_0(z)=z-\frac{A-B}{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}z^2 \quad (17)$$

Thus from (12) we have

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}f_0(z) \prec \frac{z}{1-z}, \quad (z\in\mathcal{U}) \quad (18)$$

Moreover for the function given by (17) it can be easily verified that

$$\min_{|z|\leq r}\left\{Re\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}f_0(z)\right\}=-\frac{1}{2} \quad (19)$$

this shows that the constant $\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\beta_2+(A-B)\}}$ is the best possible.

This completes the proof of theorem 1 ▪

Now we establish subordination results for the following associated subclasses, whose coefficients satisfy (11).

Putting $g(z)=\frac{z}{1-z}$, $\gamma=1$ and A = -1 in Theorem1, the class S_γ* (f,g; A,B; β) reduced to the class UCV*(β), (see Subramanian et al. [2]). We have

Corollary 1

Let the function f(z) defined by (1) be in the class UCV*(β) and suppose h(z) be in K, then.

$$\frac{\beta(1-B)-2B}{2\beta(1-B)-(1+5B)}(f*h)(z) \prec h(z), (z \in \mathcal{U}) \quad (20)$$

and

$$Re(f(z)) > -\frac{2\beta(1-B)-(1+5B)}{2[\beta(1-B)-2B]}, (z \in \mathcal{U})$$

The constant factor $\frac{\beta(1-B)-2B}{2\beta(1-B)-(1+5B)}$ in the subordination result (20) cannot be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k$, $\gamma = 0$ in Theorem1, the class S_γ^* (f,g; A,B; β) reduced to the class S_γ (f,$H_{q,s}$ (α_1); A,B; β). Where Hq,s(α_1) is Dziok-Srivastava operator (see [3]). We have

Corollary 2

Let the function f(z) defined by (1) be in the class Sγ* (f, $H_{q,s}$ (α_1); A,B; β) and suppose h(z) be in K, then.

$$\frac{[(2+\beta)(1-B)+(A-1)]\Gamma_2(\alpha_1)}{2\{[(2+\beta)(1-B)+(A-1)]\Gamma_2(\alpha_1)+(A-B)\}}(f*h)(z) \prec h(z), (z \in \mathcal{U}) \quad (21)$$

and

$$Re(f(z)) > -\frac{[(2+\beta)(1-B)+(A-1)]\Gamma_2(\alpha_1)+(A-B)}{[(2+\beta)(1-B)+(A-1)]\Gamma_2(\alpha_1)}, (z \in \mathcal{U})$$

the constant factor

$$\frac{[(2+\beta)(1-B)+(A-1)]\Gamma_2(\alpha_1)}{2\{[(2+\beta)(1-B)+(A-1)]\Gamma_2(\alpha_1)+(A-B)\}}$$

in the subordination result (21) cannot be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+L+\lambda(k-1)}{1+L}\right)^m z^k$, (λ≥0, L≥0, m ∈ N_0) in Theorem1, the class S_γ^* (f, g; A, B; β) reduced to the class S_γ^* (f, $I_{\lambda,L}^m$; A, B; β). Where $I_{\lambda,L}^m$ is Catas operator (see [7]). We have

Corollary 3. Let the function f(z) defined by (1) be in the class S_γ^* (f, $I_{\lambda,L}^m$; A, B; β) and suppose h(z) be in K, then.

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(1+\frac{\lambda}{1+L}\right)^m}{2\left\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(1+\frac{\lambda}{1+L}\right)^m+(A-B)\right\}}(f*h)(z) \prec h(z), (z \in \mathcal{U}) \quad (22)$$

and

$$Re(f(z)) > -\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(1+\frac{\lambda}{1+L}\right)^m+(A-B)}{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(1+\frac{\lambda}{1+L}\right)^m}, (z \in \mathcal{U})$$

the constant factor

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(1+\frac{\lambda}{1+L}\right)^m}{2\left\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(1+\frac{\lambda}{1+L}\right)^m+(A-B)\right\}}$$

in the subordination result (22) cannot be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty} \binom{k+\lambda-1}{\lambda} z^k$, (λ ≥ -1), (λ ≥ -1) in Theorem1, the class Sγ* (f,g; A,B; β) reduced to the class Sγ* (f, D^λ; A,B; β). Where D^λ is Ruscheweyh derivative operator (see [8]). Defined by

$$D^\lambda f(z) = \frac{z\left(z^{\lambda-1}f(z)\right)^\lambda}{\lambda!} = \frac{z}{(z+1)^{\lambda+1}} * f(z)$$

Corollary 4. Let the function f(z) defined by (1) be in the class Sγ* (f, D^λ; A,B; β). and suppose h(z) be in K, then.

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)(1+\lambda)}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)(1+\lambda)+(A-B)\}}(f*h)(z) \prec h(z), (z \in \mathcal{U}) \quad (23)$$

and

$$Re(f(z)) > -\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)(1+\lambda)+(A-B)}{[(2+\beta)(1-B)+(A-1)](1+\gamma)(1+\lambda)}, (z \in \mathcal{U})$$

the constant factor

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)(1+\lambda)}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)(1+\lambda)+(A-B)\}}$$

in the subordin-ation result (23) cannot be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+L}{1+L}\right)^m z^k$, (L ≥ 0, m ∈ N_0) in Theorem1, the class S_γ* (f,g; A,B; β) reduced to the class S_γ* (f,I_L^m; A,B; β). Where I_L^m is Cho and Kim operator (see [11]). We have

Corollary 5

Let the function f(z) defined by (1) be in the class S_γ^* (f, I_L^m; A, B; β) and suppose h(z) be in K, then.

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(\frac{2+L}{1+L}\right)^m}{2\left\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(\frac{2+L}{1+L}\right)^m+(A-B)\right\}}(f*h)(z) \prec h(z), (z \in \mathcal{U}) \quad (24)$$

and

$$Re(f(z)) > -\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(\frac{2+L}{1+L}\right)^m+(A-B)}{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(\frac{2+L}{1+L}\right)^m}, (z \in \mathcal{U})$$

the constant factor

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(\frac{2+L}{1+L}\right)^m}{2\left\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\left(\frac{2+L}{1+L}\right)^m+(A-B)\right\}}$$

in the subordination result (24) cannot be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty} k^n z^k$, (n ∈ $\mathbb{N}_0$) in Theorem1, the class Sγ* (f,g; A,B; β) reduced to the class Sγ* (f,D^n; A,B; β). Where D^n is Salagean operator (see [13]). We have

Corollary 6

Let the function f(z) defined by (1) be in the class Sγ* (f,D^n; A,B; β) and suppose h(z) be in K, then.

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)2^n}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)2^n+(A-B)\}}(f*h)(z) \prec h(z), (z \in \mathcal{U}) \quad (25)$$

and

$$Re(f(z)) > -\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)2^n+(A-B)}{[(2+\beta)(1-B)+(A-1)](1+\gamma)2^n}, (z \in \mathcal{U})$$

the constant factor

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)2^n}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)2^n+(A-B)\}}$$

in the subordination result (25) cannot be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right) z^k$, (c > -1) in Theorem1, the class Sγ* (f,g; A,B; β) reduced to the class Sγ* (f,J_c; A,B; β) . Where J_cf(z) is a Bernardi operator (see [15]), defined by

$$J_c f(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt = z + \sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)\alpha_k z^k.$$

Corollary 7

Let the function f(z) defined by (1) be in the class Sγ* (f, J_c; A,B; β) and suppose h(z) be in K, then.

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)(c+1)}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)(c+1)+(A-B)(c+2)\}}(f*h)(z) \prec h(z), (z \in \mathcal{U})$$

$$Re(f(z)) > -\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)(c+1)+(A-B)(c+2)}{[(2+\beta)(1-B)+(A-1)](1+\gamma)(c+1)}, (z \in \mathcal{U}) \quad (26)$$

and

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)(c+1)}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)(c+1)+(A-B)(c+2)\}}$$

the constant factor

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)(c+1)}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)(c+1)+(A-B)(c+2)\}}$$ in the subordination result (26) cannot be replaced by a larger one.

Putting $g(z)=z+\sum_{k=2}^{\infty}\frac{(\mu)_{k-1}}{(\lambda+1)_{k-1}}z^k$, ($\lambda > -1$, $\mu > 0$) in Theorem1, the class $S\gamma^*$ (f,g; A,B; β) reduced to the class $S\gamma^*$ (f,$I_{\lambda,\mu}$; A,B; β). Where $I_{\lambda,\mu}$ f(z) is a Choi-Saigo-Srivastava operator (see [16]), defined by

$$I_{\lambda,\mu}f(z) = z+\sum_{k=2}^{\infty}\frac{(\mu)_{k-1}}{(\lambda+1)_{k-1}}\alpha_k z^k$$

Corollary 8

Let the function f(z) defined by (1) be in the class $S\gamma^*$ (f,$I_{\lambda,\mu}$; A,B; β and suppose h(z) be in K, then.

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\mu}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\mu+(A-B)(\lambda+1)\}}(f*h)(z) \prec h(z), (z\in\mathcal{U}) \quad (27)$$

and

$$Re(f(z)) > -\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\mu+(A-B)(\lambda+1)}{[(2+\beta)(1-B)+(A-1)](1+\gamma)\mu}, (z\in\mathcal{U})$$

the constant factor

$$\frac{[(2+\beta)(1-B)+(A-1)](1+\gamma)\mu}{2\{[(2+\beta)(1-B)+(A-1)](1+\gamma)\mu+(A-B)(\lambda+1)\}}$$

in the subordination result (27) cannot be replaced by a larger one.

We have the following interesting relationships with some of the special function classes for suitable choices of parameters, which were investigated recently:

(i). For $g(z)=\frac{z}{1-z}, \gamma=0, A=1-2\alpha\ (0\le \alpha<1), B=-1\ and\ \beta=$ in Theorem1. We obtain the result obtained by Frasin [17, Corollary 2.3].

(ii) For $g(z)=\frac{z}{1-z}, \gamma=0, A=1,\ B=-1$ and β=0 in Theorem1. We obtain the result obtained by Frasin [17, Corollary 2.4].

(iii). For $g(z)=\frac{z}{1-z}, \gamma=A=1, B=-1$ and β=0 in Theorem1. We obtain the result obtained by Frasin [17, Corollary 2.7].

(iv). For $g(z)=\frac{z}{1-z}, \gamma=1, A=1-2\alpha\ (0\le$ α<1),B=-1 and β=0 in Theorem1. We obtain the result obtained by Frasin [17, Corollary 2.6].

(v). For $g(z)=\frac{z}{(1-z)^2}, \gamma=0, A=1-2\alpha$ (0≤α<1), B=-1 and β=0 in Theorem1. We obtain the result obtained by Frasin [17, Corollary 2.5].

(vi) For $g(z)=z+\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k\ (c\ne 0,-1,-2,\dots\dots)\ and\ \gamma=0,\ A=1-2\alpha\ (0\le\alpha<1), B=-1$ in Theorem1. We obtain the result obtained by Frasin [17, Theorem 2.1].

(vii) For $A=1-2\alpha(0\le\alpha<1)\ and\ B=$ -1 in Theorem1. We obtain the result obtained by Aouf et. Al. [18, Theorem 1].

(viii). For $g(z)=z+\sum_{k=2}^{\infty}\mu_k z^k, \gamma=0, A=1-2\alpha\ (0\le\alpha<1) and\ B=-1$ in Theorem1. We obtain the result obtained by Raina and Bansal [19, Theorem 5.2]

(ix). For β=0 in Theorem1. We obtain the result obtained by Ashwah, Aouf and Drbuk [20, Theorem1].

References

[1] Attiya A.A.,"On some application of a subordination theorems", J. Math. Anal. Appl. 311, 2005, pp. 489-494.

[2] Subramanian K.G. Murugusundaramorthy G., Balasubrahma P., Silverman H.,"subclasses of unifrmly convex and uniformly starlike functions", Math. Japon. 42, No. 3, 1995, pp. 517-522.

[3] Dziok J., Srivastava H.M.,"Classes of analytic functions with the generalized hypergeometric functions", Appl. Math. Comput., 103, 1999, pp. 1-13.

[4] Srivastava H.M., Attiya A.A.,"Some subordination results associated with certain subclass of analytic functions", J. Inequal. Pure Appl. Math. 5, No. 4, Art 86 (2004), pp. 1-6.

[5] Goodman A.W.,"On uniformly convex functions" , Ann. Polon. Math. 56, 1991, pp. 87-92.

[6] Goodman A.W.,"On uniformly starlike functions", J. Math. Anal. Appl. 155, 1991, pp. 364-370.

[7] Catas A.,"On certain classes of p-valent functions defined by multiplier transformations", Proceeding of the International Symposium on Geometric Function Theory and Application, GFTA 2007 Proceedings, Istanbul, Turkey, August 2007, pp. 20-24.

[8] Ruscheweyh St.,"New Criteria for univalent functions", Proc. Amer. Math. Soc., 49, 1975, pp. 109-115.

[9] Ronning E.,"On starlike functions associated with parabolic regions", Ann. Univ. Mariae-Curie_Sklodowska, Sect. A45 1991, pp. 117-122.

[10] Ronning E.,"Uniformly convex functions and a corresponding class of starlike functions", Proc. Amer. Math. Soc. 118, 1993, pp. 189-196.

[11] Cho N.E., Kim T.H.,"Multiplier transformations and strongly close to convex functions", Bull. Korean. Math. Soc., 40, 2003, No. 3, pp 399-410.

[12] Miller S.S., Mocanu P.Y.,"Differential subordinance theory and applications, Series on Monographs", Pure. And Appl. Math., (2000), No. 255 Marcel Dekker, Inc., New York.

[13] Salagean G.S.,"Subclasses of univalent functions", Lecture notes in Math., 1013, Springer Verlage, Berlin, 1983, pp. 362-372.

[14] Wilf H.S.,"Subordinating factor sequence for convex maps of the unit circle", Proc. Amer. Math. Soc., 12(1961), pp. 689-693.

[15] Bernardi S.D.,"Convex and starlike functions", Trans. Amer. Math. Soc., 135, 1969, pp. 429-446.

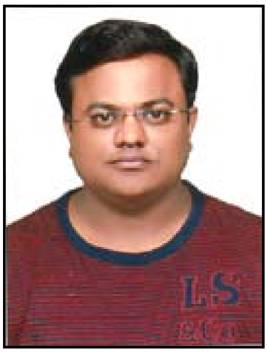
[16] Choi J.H., Saigo M., Srivastava H.M.,"Some inclusion properties of a certain family of integral operator", J. Math. Anal. Appl. 292, 2004, pp. 432-445.

[17] Frasin B.A., "Subordination results for a class of analytic functions defined by a linear operator", J. Inequal. Pure. Appl. Math. 7, No. 4, Art 134, 2006, pp. 1-7.

[18] Aouf M.K., El-Ashwah R.M., El-deeb S.M.,"Subordina-tion results for certain subclasses of uniformly starlike and convex functions defined by convolution", European J. Pure. Appl. Math., 3, 2010, No. 5, pp. 903-917.

[19] Raina R.K., Deepak Bansal,"Some properties of a new class of analytic functions defined in terms of a Hadamard product", J. Inequal. Pure. Appl. Math. 9, No. 1, Art 22 (2008), pp. 1-9.

[20] El-Ashwah R.M., Aouf M.K., Drbuk M.E.,"Subordination properties for new certain subclass of univalent functions", Electronic J. Math. Anal. And Appl., Vol. 2(2) (2014), pp. 238-247.



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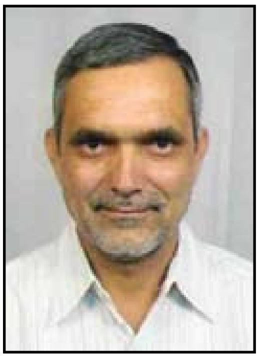

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