Pivoting Strategies and Their Applications for Solving Linear Systems

Ashu Vij
Dept. of Mathematics, DAV College, Amritsar, Punjab, India

Abstract

While using elimination process for solving linear systems, finite precision arithmetic is used. In some cases, this leads to completely erroneous answers. To avoid this pivoting strategies are used. In this paper different pivoting strategies are discussed for solving linear systems. Thereby a numerical problem is illustrated to support the results.

Keywords

Linear Systems, Pivoting Strategies, Elimination Process

I. Introduction

The system of linear equations which describe the relationship between system variables occur quite often in the field of science and engineering. In simple cases, there may be two or three variables; in complex cases e.g. in a linear model of economy of a country, there may be several hundred variables. The system of linear equations also arises in connection with many problems of numerical analysis. So it is important to have rapid and accurate methods for solving linear equation. There are a lot of numerical methods available for solution of linear systems. Each one out of these methods has its own advantages. The most common of these methods is Elimination Method. In this method, the given linear system is transformed into an equivalent system which is in upper triangular form; this new form can be solved easily by the process of back substitution [1-3].

When transforming linear systems of equations to an upper triangular form, one shall use one or more of the following elementary operations at every step:

- Multiplication of an equation by a constant.
- Subtraction from one equation some multiple of other equation.
- Interchanging of two equations.

A. Pivoting

In elimination process, a pivot element is very small as compared to the elements in its column which have to be eliminated; the corresponding multipliers used at that stage will be greater than 1 in magnitude. The use of large multipliers in elimination and back substitution process lead to magnification of round off errors. Also method fails if the pivot element at any stage becomes zero. The built up of round off errors or getting the pivot element as a zero may be avoided by rearranging the remaining rows. This strategy is called pivoting [5]. This work in addition to the elimination procedure is quite tidy. For this, at the beginning of ith elimination step, one may search for a non zero coefficient for xi in the equations i, i+1,----, n and if it is found in some equation j > i, interchange equations j and i. this freedom of interchanging helps in reducing rounding errors while doing calculations with finite precision floating arithmetic.

B. Numerical Problem

The solution of system of equations

$0.1410 \times 10^{-2} X_1 + 0.4004 \times 10^{-1} X_2 = 0.1142 \times 10^{-1}$

$0.2000 \times 10^{-0} X_1 + 0.4912 \times 10^{1} X_2 = 0.1428 \times 10^{1}$

is $x_1 = 1.000$ and $x_2 = 0.2500$, using four decimal floating arithmetic with two digit exponent [4].

1. Let us try to solve above linear system, taking the first equation as pivotal equation.

The multiplier is $0.2000 \times 10^0 / 0.1410 \times 10^{-2} = 0.1418 \times 10^3$

new $a_{22} = 0.4912 \times 10^1 - (0.4004 \times 10^{-1})(0.1418 \times 10^3)$

$= -0.7657$

New $b_2 = 0.1428 \times 10^1 - (0.1142 \times 10^{-1})(0.1418 \times 10^3)$

$= -0.1914$

So, $x^2 = -0.1914 / -0.7657 = 0.2410$

Hence, from first equation, on substituting value of x_2, we have,

$x_1 = 0.1256 \times 10^1$

The solution obtained by above procedure is too far from exact solution.

2. Let us try to solve above linear system , picking the second equation as pivot equation.

For convenience, rewrite the above system as

$0.2000 \times 10^{-0} X_1 + 0.4912 \times 10^1 X_2 = 0.1428 \times 10^1$

$0.1410 \times 10^{-2} X_1 + 0.4004 \times 10^{-1} X_2 = 0.1142 \times 10^{-1}$

The multiplier is $0.1410 \times 10^{-2} / 0.2000 \times 10^0 = 0.7050 \times 10^{-2}$

New $a_{22} = 0.4004 \times 10^{-1} - (0.4912 \times 10^1)(0.7050 \times 10^{-2})$

$= 0.5410 \times 10^{-2}$

new $b_2 = 0.1142 \times 10^{-1} - (0.1428 \times 10^1)(0.7050 \times 10^{-2})$

$= 0.1353 \times 10^{-2}$

So, $x_2 = 0.1353 \times 10^{-2} / 0.5410 \times 10^{-2} = 0.2501$

Hence, from first equation, on substituting value of x1, we have,

$x_1 = 0.9975$

3. Let us try to solve above system of equation by making a_{22} as first pivot element.

So, rewrite the given equations as

$0.4912 \times 10^1 X_2 + 0.2000 \times 10^{-0} X_1 = 0.1428 \times 10^1$

$0.4004 \times 10^{-1} X_2 + 0.1410 \times 10^{-2} X_1 = 0.1142 \times 10^{-1}$

The multiplier is $0.4004 \times 10^{-1} / 0.4912 \times 10^1 = 0.8151 \times 10^{-2}$

So new second equation becomes $-0.2202 \times 10^{-3} X_1 = -0.2196 \times 10^{-3}$

So, $X_1 = 0.9973$

Hence, from first equation, on substituting value of X_1, we have,

$X_2 = 0.2501$

II. Results and Discussion

The solution of above systems by three different strategies clearly indicates that procedure 2 and 3 is very close to exact solution. In procedure 2, we search the first column for largest element in absolute value and interchange the first equation with that having the largest element in absolute value. The new first pivot element is largest in absolute value than any other element beneath it in its column. This procedure is repeated till we reach the last equation. It is called partial pivoting. The strategy adopted in 3 is total pivoting or complete pivoting. In this procedure, we search the coefficient matrix for largest element in absolute value and bring it as first pivot. This procedure not only requires an interchange of equations but also an interchange of position of variables. This procedure is a little bit complicated if we have to solve a very large system. In procedure 1, the solution obtained is too far from an exact solution. In procedure 1, we have considered first equation as

ISSN : 2348-0033 (Online) | ISSN : 2249-4944 (Print)

pivot equation without analyzing the coefficient matrix i.e without adopting any strategy. The failure of this procedure is due to the fact that the pivot element is very small.

III. Conclusion

From above discussion, it is very much clear that how various pivoting strategies affect the accuracy of computed equations. Of the above discussed strategies, total pivoting is much more expensive in terms of labor required for computation especially for solving very large systems. But this does not improve the solution very much as compared to that calculated by partial pivoting. On comparing solutions compared by all three strategies, an important fact to be observed is that the multiplier in strategies 2 and 3 is very small as compared with that in strategy 1 because large multipliers in elimination procedure lead to magnification in round off errors.

References

[1] Wilkinson, J.H."Error analysis of direct methods of matrix inversion", J. ACM 8, 3 (July 1961), pp. 281-330.
[2] Wilkinson, J.H.,"Rounding Errors in Algebraic Processes", Prentice-Hall, Englewood
[3] L. V. Foster,"Gaussian elimination with partial pivoting can fail in practice", SIAM J. Matrix Anal. Appl., 15 (1994), pp. 1354–1362.
[4] De Boor, Elementary numerical analysis, McGraw-Hill Book Company,(1981) pp. 150-160
[5] Hildebrand, F.B.,"Introduction to numerical analysis", McGraw-Hill, NewYork, 1956.

Ashu Vij received his B.Sc. degree in computer science from D.A.V. College, Jalandhar in 2001, M.Sc. Degree in Mathematics from D. A.V. College, Jalandhar in 2003. He is teaching as assistant professor in P.G. department of mathematics, D. A.V. College, Amritsar. His teaching experience is 11 years and areas of interest are numerical analysis and operation research.